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# Convergence of Mimetic Finite Difference Method for Diffusion Problems on Polyhedral Meshes

**Franco Brezzi**

University of Pavia, Italy  
brezzi@imati.cnr.it

**Konstantin Lipnikov**

**Mikhail Shashkov**

Los Alamos National Laboratory, USA  
lipnikov@lanl.gov, shashkov@lanl.gov

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# Motivation

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- Sources of polyhedral meshes:
  - meshing of complex geometries
  - adaptive mesh refinement methods
  - multi-block meshes (e.g., non-matching meshes)
  - mesh reconnection methods (e.g., ALE methods)

# Motivation

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- The MFD method gives a rich family of discretization schemes with equivalent properties.
- On simplicial meshes, this family includes schemes appearing in mixed finite element methods.
- The MFD method can be formally designed on meshes with non-convex and degenerate elements.

# Mimetic finite difference method

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$$\vec{F} = -K \operatorname{grad} p, \quad \operatorname{div} \vec{F} = b, \quad \operatorname{div} = -(K \operatorname{grad})^*, \quad \operatorname{Null}(\operatorname{grad}) = \operatorname{const}$$



*MFD*

$$F^h = -\mathcal{G} p^h, \quad \mathcal{DIV} F^h = b^h, \quad \mathcal{DIV} = -\mathcal{G}^*, \quad \operatorname{Null}(\mathcal{G}) = \operatorname{const}$$

# Mimetic finite difference method

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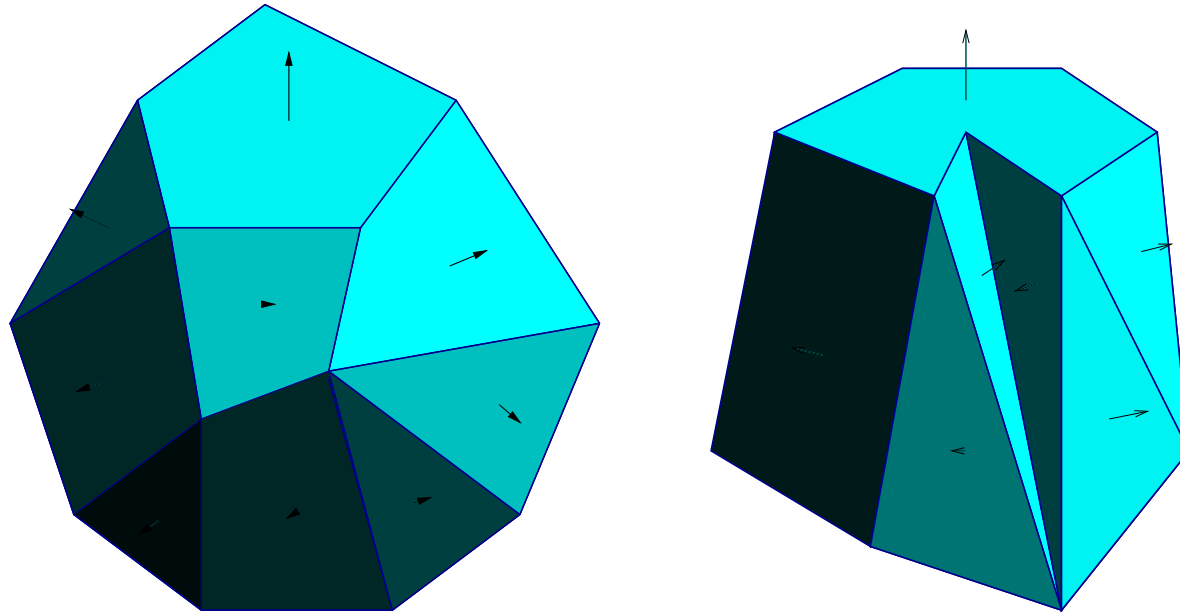
## Four-step methodology:

1. Define degrees of freedom for  $p^h \in Q_h$  and  $F^h \in X_h$
2. Equip discrete spaces with scalar products
3. Discretize the divergence operator
4. **Derive** the discrete flux operator from discrete Green's formula

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# Mimetic finite difference method

**Step 1.** Define degrees of freedom for  $p^h \in Q_h$  and  $F^h \in X_h$

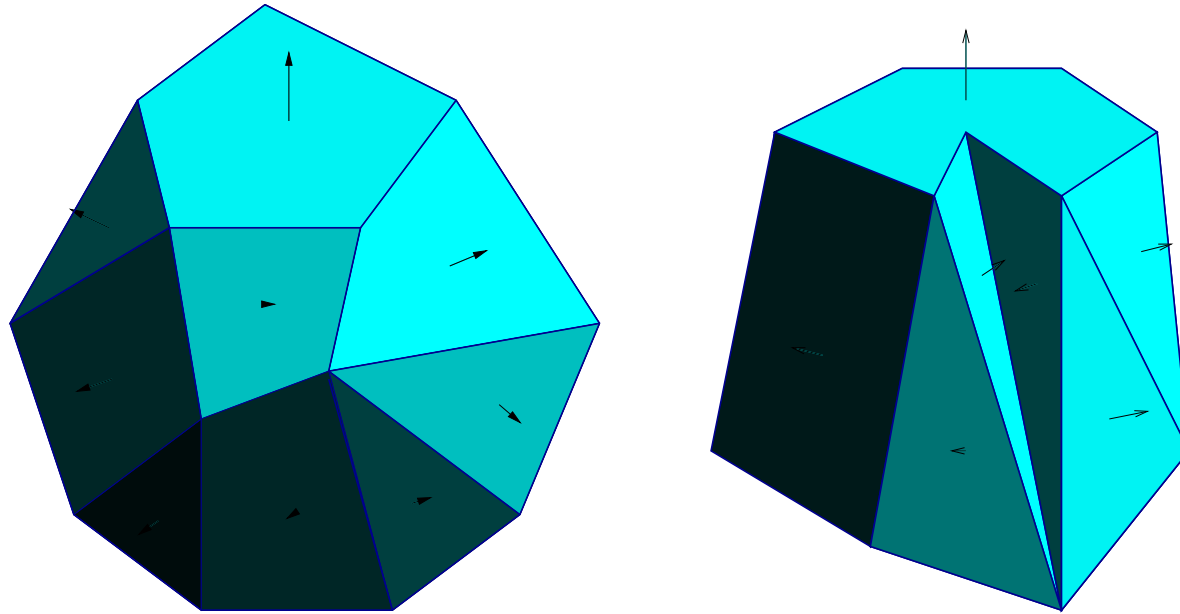


■  $p^h$  is constant on each polyhedron

■  $(p^h)_E$  is the degree of freedom associated with polyhedron  $E$

# Mimetic finite difference method

**Step 1.** Define degrees of freedom for  $p^h \in Q_h$  and  $F^h \in X_h$



■  $F^h$  is constant on each mesh face

■  $(F^h)_f$  is the normal flux component associated with mesh face  $f$

# Mimetic finite difference method

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## Step 2. Equip discrete spaces with scalar products

$$\blacksquare [p^h, q^h]_Q = \sum_{E \in \Omega_h} (p^h)_E (q^h)_E |E| \approx \int_{\Omega} p q dV$$

$$\blacksquare [F^h, G^h]_X = \sum_{E \in \Omega_h} [F^h, G^h]_E \approx \int_{\Omega} \vec{F} \cdot \vec{G} dV$$

where

$$[F^h, G^h]_E = \sum_{i,j=1}^{k_E} M_{E,i,j} (F^h)_{f_i} (G^h)_{f_j}$$

and  $M_E$  is an SPD matrix (it is **not** unique!).

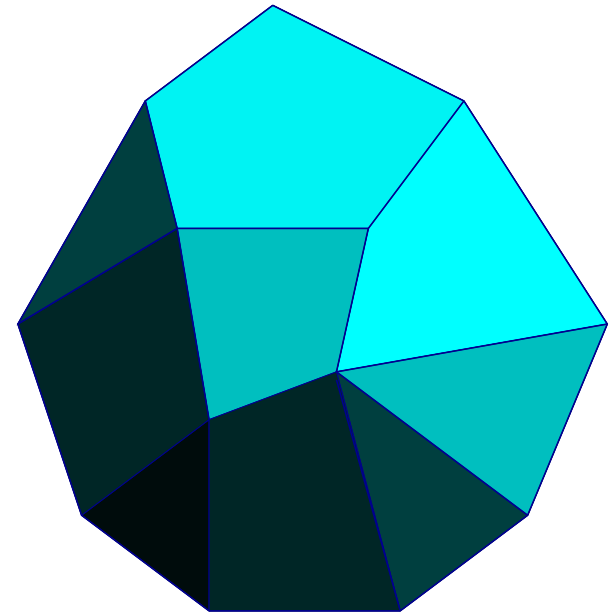
# Mimetic finite difference method

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## Steps 3. Discretize the divergence operator

The divergence theorem

$$\operatorname{div} \vec{F} = \lim_{|E| \rightarrow 0} \frac{1}{|E|} \oint_{\partial E} \vec{F} \cdot \vec{n} \, dx$$



implies

$$\left( \mathcal{DIV} \mathbf{F}^h \right)_E = \frac{1}{|E|} \sum_{f \in \partial E} (\mathbf{F}^h)_f |f|$$

# Mimetic finite difference method

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## Steps 4. **Derive** the discrete flux operator

- The continuous operators satisfy Green's formula

$$\int_{\Omega} \vec{F} \cdot K^{-1}(K \operatorname{grad} p) \, dx = - \int_{\Omega} p \operatorname{div} \vec{F} \, dx.$$

- We enforce that the discrete operators satisfy discrete Green's formula

$$[\mathbf{F}^h, \mathcal{G} \mathbf{p}^h]_X = -[\mathbf{p}^h, \mathcal{DIV} \mathbf{F}^h]_Q \quad \forall \mathbf{p}^h \in Q_h \quad \forall \mathbf{F}^h \in X_h.$$

# Meshes covered by the theory

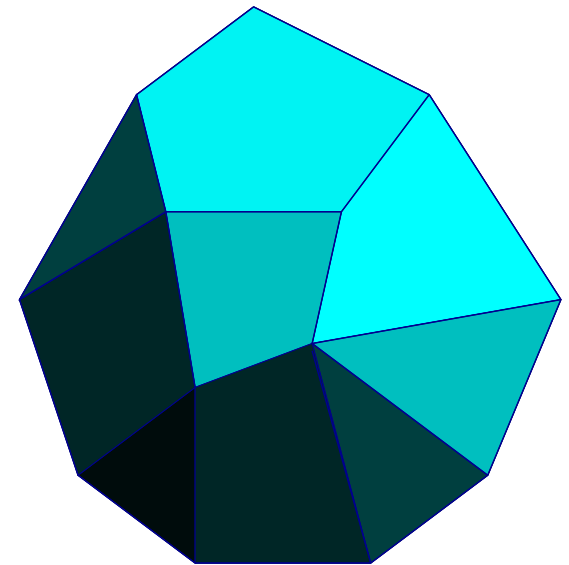
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## Our analysis forbid:

- anisotropic (stretched) elements
- stretched faces
- small 2D angles

## Our analysis allow:

- convex elements



# Meshes covered by the theory

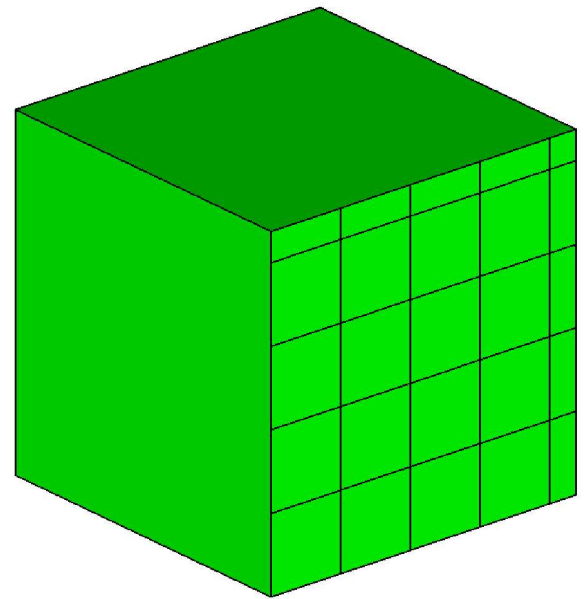
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# Meshes covered by the theory

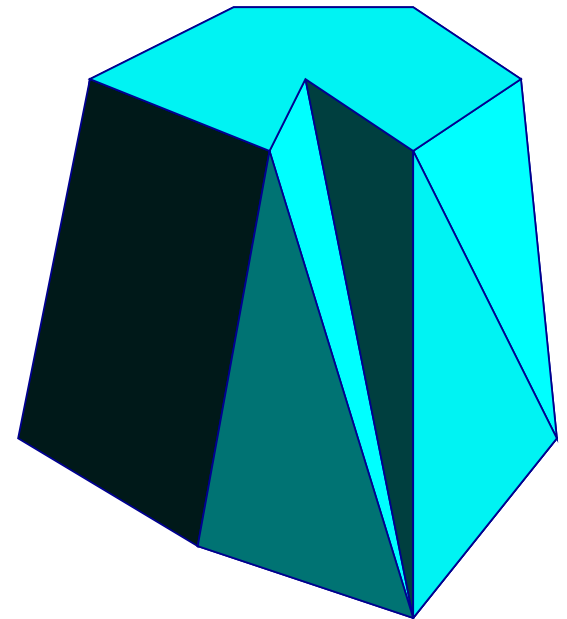
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## Our analysis forbid:

- anisotropic (stretched) elements
- stretched faces
- small 2D angles

## Our analysis allow:

- convex elements
- degenerate elements
- non-convex elements



# Key theoretical assumptions

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- For every element  $E$  and for every  $\mathbf{G}^h \in X_h$ , there are two positive constants  $s_*$  and  $S^*$  s.t.

$$s_* |E| \sum_{f \in \partial E} (\mathbf{G}^h)_f^2 \leq [\mathbf{G}^h, \mathbf{G}^h]_E \leq S^* |E| \sum_{f \in \partial E} (\mathbf{G}^h)_f^2$$

- matrix  $M_E$  is spectrally equivalent to the scalar matrix  $|E|I$ .

# Key theoretical assumptions

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- For every element  $E$  with the center of gravity at the origin and every  $\mathbf{G}^h \in X_h$ , we have

$$[(K \nabla q^1)^I, \mathbf{G}^h]_E = \int_{\partial E} q^1 \mathbf{G}^h \, dx$$

where

$$q^1 = x, \quad q^1 = y \quad \text{and} \quad q^1 = z.$$

# Key theoretical assumptions

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- For every element  $E$  with the center of gravity at the origin and every  $\mathbf{G}^h \in X_h$ , we have

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where

$$q^1 = x, \quad q^1 = y \quad \text{and} \quad q^1 = z.$$

- discrete flux operator  $\mathbf{G}^h$  is exact for linear distribution of pressure.

# Estimate for the vector variable

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- **Theorem.** Let  $(p, \vec{F})$  be the continuous solution,  $(p^h, \mathbf{F}^h)$  be the discrete solution and  $\mathbf{F}^I$  be the interpolant of  $\vec{F}$ . Then

$$|||\mathbf{F}^I - \mathbf{F}^h|||_X \leq C^* h \|p\|_{H^2(\Omega)}$$

where

$$h = \max_{E \in \Omega_h} h_E.$$

# Estimates for the scalar variable

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- **Theorem.** Let  $(p, \vec{F})$  be the continuous solution,  $(\mathbf{p}^h, \mathbf{F}^h)$  be the discrete solution and  $\mathbf{p}^I$  be the interpolant of  $p$ . For *convex* domain  $\Omega$ , we get

$$|||\mathbf{p}^I - \mathbf{p}^h|||_Q \leq C^* h \left( \|p\|_{H^2(\Omega)} + \|b\|_{H^1(\Omega)} \right).$$

With a few additional assumptions, we get

$$|||\mathbf{p}^I - \mathbf{p}^h|||_Q \leq C^* h^2 \left( \|p\|_{H^2(\Omega)} + \|b\|_{H^1(\Omega)} \right).$$

# Computing matrix $M_E$

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- matrix  $M_E$  has  $k(k + 1)/2$  unknown entries:

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{12} & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{pmatrix} \quad \text{for } k = 4 \quad (\text{tetrahedron})$$

- the key theoretical assumptions result in a linear system

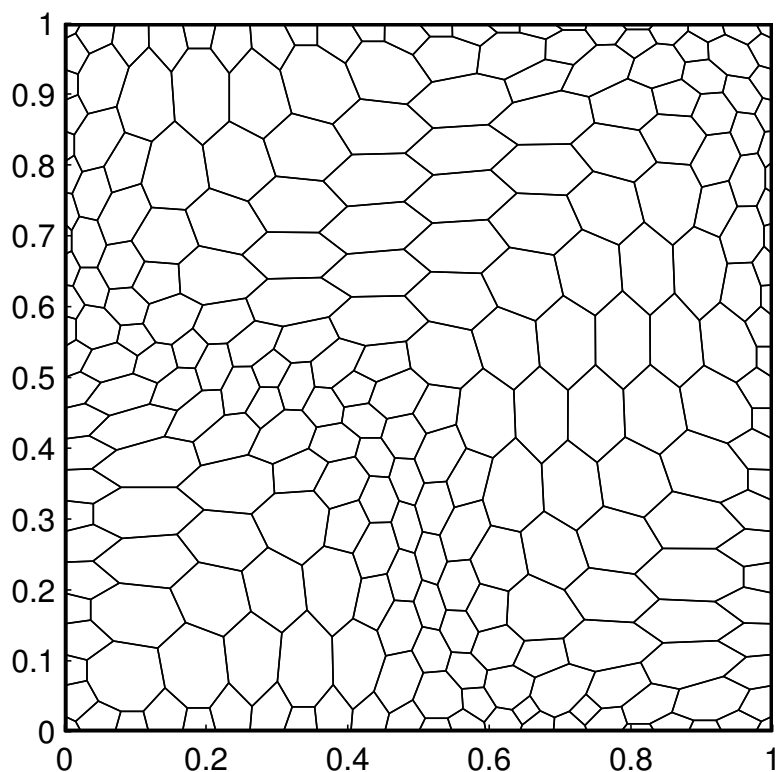
$$A M_E = C$$

- the solution matrix  $M_E$  is not unique !!!

# Polygonal meshes

Let  $p(x, y) = x^3 y^2 + x \sin(2\pi x y) \sin(2\pi y)$  and

$$K(x, y) = \begin{pmatrix} (x+1)^2 + y^2 & -xy \\ -xy & (x+1)^2 \end{pmatrix}.$$

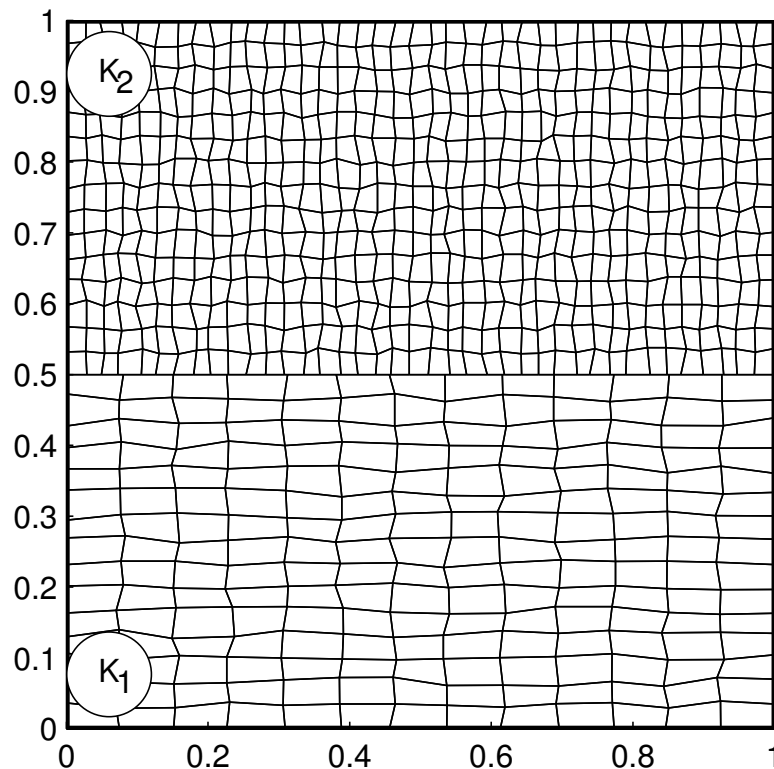


$1/h$	$    \mathbf{p}^I - \mathbf{p}^h    _Q$	$    \mathbf{F}^I - \mathbf{F}^h    _X$
16	5.17e-2	7.38e-1
32	1.18e-2	2.44e-1
64	2.76e-3	8.45e-2
128	6.65e-4	2.89e-2
rate	2.09	1.56

# Random non-matching meshes

Let  $a = b = c = 1$ ,  $K_1 = 10$ ,  $K_2 = 1$  and  $m = 3$  in

$$p(x, y) = \begin{cases} a + bx + cy^m, & y < 0.5, \\ a + c \frac{K_2 - K_1}{2^m K_2} + bx + c \frac{K_1}{K_2} y^m, & y \geq 0.5. \end{cases}$$

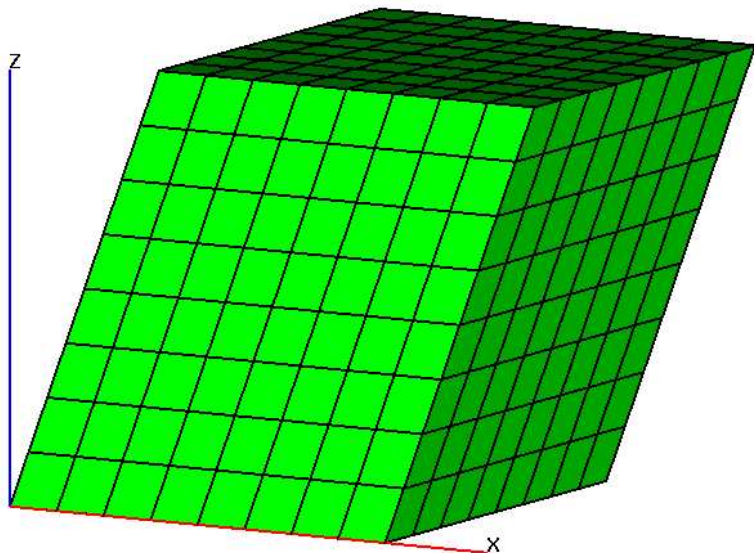


# cells	$   p^I - p^h   _Q$	$   F^I - F^h   _X$
780	1.01e-2	1.12e-1
3286	2.36e-3	4.72e-2
13482	5.73e-4	2.24e-2
54610	1.41e-4	1.09e-2
rate	2.01	1.09

# Polyhedral meshes

Let  $p(x, y) = x^3 y^2 z + x \sin(2\pi xy) \sin(2\pi yz) \sin(2\pi z)$  and

$$K(x, y, z) = \begin{pmatrix} 1 + y^2 + z^2 & -xy & -xz \\ -xy & 1 + x^2 + z^2 & -yz \\ -xz & -yz & 1 + x^2 + y^2 \end{pmatrix}.$$



$1/h$	$    \mathbf{p}^I - \mathbf{p}^h    _Q$	$    \mathbf{F}^I - \mathbf{F}^h    _X$
8	3.83e-2	5.35e-1
16	1.10e-2	1.43e-1
32	2.86e-3	3.58e-2
64	7.21e-4	8.94e-3
rate	1.91	1.97

# Conclusion

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- We developed a **new** methodology for the design and the analysis of the MFD method.
- We proved **optimal** convergence estimates.
- Possible extensions:
  - $h^2$ -curved faces (smooth meshes)
  - problems with a lack of elliptic regularity
  - other PDEs (Maxwell, linear elasticity, etc)
  - strongly curved faces